Upper hemi-continuity

- Best-response correspondences have to be upper hemi-continuous for Kakutani's fixed-point theorem to work
- Upper hemi-continuity requires that:
  - The correspondence have a closed graph (the graph does contain its bounds), i.e.

  $f: A \to Y$ has a closed graph if for any two sequences $x^m \to x \in A$ and $y^m \to y$, with $x^m \in A$ and $y^m \in f(x^m)$ for every m, we have $y \in f(x)$
  - The images of compact sets are bounded i.e.

  if for every compact set $B \subset A$ the set $f(B)$ is bounded
- The first condition is enough whenever the range of correspondence is compact, which is the case with Nash Theorem

Normal-Form Games: Applications

- So far we've analyzed trivial games with a small number of strategies
- We will now apply IEDS and NE concepts to Normal-Form Games with infinitely many strategies
  - Divide a Benjamin
  - Second-price auction
  - First-price auction
  - Price-setting duopoly (Bertrand model)

Divide a Benjamin

- Two players select a real number between 0 and 100
- If the two numbers add up to 100 or less, each player gets the payoff = the selected number
- If the two numbers add up to more than 100, each player gets nothing
- Task: Secretly select a number, your opponent will be selected randomly.
- Analysis: The set of NE in this game is infinite (all pairs of numbers which sum up to exactly 100). Only one strategy (0) is weakly dominated.
- Yet people can predict quite well how this game will be played in reality

Second-Price Auction

- There is one object for sale
- There are 9 players, with valuations of an object equal to their index ($v_i = i$)
- Players submit bids b_i
- The player who submits the highest bid is the winner (if tied, the higher-index player is the winner)
- The winner pays the price equal to the second-highest bid (b_s), so his payoff is $v_i – b_s$
- All other players receive 0 payoffs
- Analysis: Notice that bidding anything else than own true valuation is weakly dominated
- Yet, there are some strange NE, e.g. one in which the winner is the player with the lowest valuation (b_1=10, b_2=b_3=..=b_9=0)

First-Price Auction

- Same as above, except...
- The winner pays the price equal to her own bid, so her payoff is $v_i – b_i$
- Analysis: Notice that bidding above or at own valuation is weakly dominated
- In all NE the highest-valuation player (9) wins and gets a payoff between 0 and 1

Price-setting duopoly

- In the model introduced by Bertrand (1883), two sellers (players) choose and post prices simultaneously
- The consumers (not players) automatically buy from the lower-price seller, according to the demand curve
- If prices are the same, the demand is split 50-50 between the sellers
- Let us consider a version with
  - costs equal to 0
  - demand curve: $Q = 80 - 10*P$
  - $S_1 = S_2 = \{0, 1, 2, 3, 4\}$

Discrete version

Try solving by IEDS and find NE

		Price of firm 2 (P_2)				
Price of firm 1 ($\mathbf{P_1}$)		*4*	*3*	*2*	*1*	*0*
	4	**80**, *80*	**0**, *150*	**0**, *120*	**0**, *70*	**0**, *0*
	3	**150**, *0*	**75**, *75*	**0**, *120*	**0**, *70*	**0**, *0*
	2	**120**, *0*	**120**, *0*	**60**, *60*	**0**, *70*	**0**, *0*
	1	**70**, *0*	**70**, *0*	**70**, *0*	**35**, *35*	**0**, *0*
	0	**0**, *0*	**0**, *0*	**0**, *0*	**0**, *0*	**0**, *0*

Continuous version

- Let us consider a more general version
  - marginal costs equal to $c < 1/4$
  - (inverse) demand curve: $P = 1 - Q$
  - $S_1 = S_2 = [0, +\infty)$
- We will now specify payoff functions, state and graph best response correspondences

Best-response correspondences

- The profit (payoff) of firm *i* is:
  - $\Pi_i = (p_i - c)q_i$
  - $q_i = 0$ if $p_i > p_j$
  - $q_i = 1 - p_i$ if $p_i < p_j$
  - $q_i = (1 - p_i)/2$ if $p_i = p_j$
- And the best response is:
  - $p_i = p^M$ if $p_j > p^M$ (monopoly price),
  - $p_i = p_j - \varepsilon$ if $c < p_j \leq p^M$
  - $p_i \geq c$ if $p_j = c$
  - $p_i > p_j$ if $p_j < c$

Robustness

- NE = {c,c} – is this a paradox?
- When costs differ, we have a monopoly
- But the best response always the same: undercut the opponent, unless it would mean selling below cost
- BR different if there are capacity constraints
- Lowest-price guarantees – change the best response, undercutting no longer optimal